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An inequality involving Beta-functions

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This report deals with a question put by the Statistical Dept. It concerns an inequality in which complete and incomplete Beta-functions occur. The result is as follows.

If a and b are positive, then for $0 < g \leq \frac{1}{2}$ one has

$$(1)\quad \frac{\int_0^g \int_0^g (xy)^{a-1}(1-x-y)^{b-1}dxdy}{B(a,b)B(a,a+b)} < \left\{ \frac{\int_0^g x^{a-1}(1-x)^{a+b-1}dx}{B(a,a+b)} \right\}$$

Proof. First of all we deal with the particular case $g = \frac{1}{2}$. We denote the left hand member of (1) by L (g). Let T be the triangle bounded by the lines $x = y = 0$, $x+y = 1$ and T' the triangle bounded by the lines $x = \frac{1}{2}$, $y = 0$, $x+y = 1$. Then for reasons of symmetry we have

$$L\left(\tfrac{1}{2}\right) = \frac{1}{B(a,b)B(a,a+b)} \cdot \left\{ \iint_T - 2 \iint_{T'} \right\} .$$

If in both integrals of the last member we apply the substitution $x = x$, $y = u(1-x)$ we get

$$L\left(\tfrac{1}{2}\right) = \frac{1}{B(a,b)B(a,a+b)} \cdot \left\{ \int_0^1 x^{a-1}(1-x)^{a+b-1}dx. \int_0^1 u^{a-1}(1-u)^{b-1}\, du \right.$$

$$\left. -2 \int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{a+b-1}\, dx. \int_0^1 u^{a-1}(1-u)^{b-1}\, du \right\}$$

$$= 1 - \frac{2}{B(a,a+b)} \int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{a+b-1}\, dx.$$

Consequently

$$L\left(\tfrac{1}{2}\right) < \left\{ 1 - \frac{1}{B(a,a+b)} \int_{\frac{1}{2}}^1 x^{a-1}(1-x)^{a+b-1}\, dx \right\}^2$$

$$= \left\{ \frac{1}{B(a,a+b)} \int_0^{\frac{1}{2}} x^{a-1}\, (1-x)^{a+b-1}\, dx \right\}^2 ,$$

which proves the result for $g = \frac{1}{2}$.

Next we deduce from this result that (1) also holds for $0 < g < \frac{1}{2}$. We put

$$\frac{B(a,b)}{B(a,a+b)} = c ,$$

$$c \left\{ \int_0^g x^{a-1} (1-x)^{a+b-1} dx \right\}^2 - \int_0^g \int_0^g (xy)^{a-1} (1-x-y)^{b-1} dx\, dy =$$

$$= \varphi(g).$$

We shall prove that there exists a point g_0 with $0 < g_0 < \frac{1}{2}$, such that $\varphi(g)$ is steadily increasing for $0 < g < g_0$ and steadily decreasing for $g_0 < g < \frac{1}{2}$. The relation (1) will then be proved completely.

We note that on account of the logarithmic convexity of the Γ - function we have

$$c = \frac{B(a,b)}{B(a,a+b)} = \frac{\Gamma(b)}{\Gamma(a+b)} \cdot \frac{\Gamma(2a+b)}{\Gamma(a+b)} > 1.$$

Differentiating $\varphi(g)$ with respect to g we get

$$\varphi'(g) = 2c\, g^{a-1}(1-g)^{a+b-1} \int_0^g x^{a-1} (1-x)^{a+b-1} dx$$

$$-2 \int_0^g (gx)^{a-1} (1-g-x)^{b-1} dx$$

$$= 2g^{a-1} (1-g)^{a+b-1} \left\{ c \int_0^g x^{a-1}(1-x)^{a+b-1} - \int_0^{\frac{g}{1-g}} x^{a-1}(1-x)^{b-1} \right.$$

$$= 2g^{a-1} (1-g)^{a+b-1} \cdot \varphi_1(g) , \quad \text{say.}$$

Next differentiating $\varphi_1(g)$ we find

$$\varphi_1'(g) = c . g^{a-1}(1-g)^{a+b-1} - \frac{1}{(1-g)^2} \left(\frac{g}{1-g}\right)^{a-1} \left(\frac{1-2g}{1-g}\right)^{b-1}$$

$$= g^{a-1}(1-g)^{-(a+b)} \left\{ c(1-g)^{2(a+b)-1} - (1-2g)^{b-1} \right\}$$

$$= g^{a-1}(1-g)^{-(a+b)} \quad \varphi_2(g), \text{ say.}$$

Clearly

$$(2) \quad \begin{cases} \varphi_1(0) = 0, \\ \varphi_1(\frac{1}{2}) < c \int_0^1 x^{a-1}(1-x)^{a+b-1} dx - \int_0^1 x^{a-1}(1-x)^{b-1} dx \\ \qquad = c\, B(a,a+b) - B(a,b) = 0, \\ \varphi_2(0) = c-1 > 0 , \quad \varphi_2(\frac{1}{2}) > 0. \end{cases}$$

Further we have $\varphi_2(g) = 0$ if and only if

$$1-2g = \sqrt[b-1]{c.} \quad (1-g)^{\frac{2(a+b)-1}{b-1}} \quad \text{in the case } b \neq 1,$$

$$1 = c. \ (1-g)^{2a+1} \quad \text{in the case } b = 1.$$

Since for fixed real $\alpha \neq 1$ the function $f(t) = t^{\alpha}$ is either a convex or a concave function for $t > 0$, it follows that $\varphi_2(g) = 0$ for at most two values of g.

Since $\varphi_2(g)$ is the derivative of $\varphi_1(g)$, apart from a positive factor for $g \neq 0$, it follows from the above result and the relations (2) that $\varphi_1(g)$ is equal to zero for $g = 0$, positive for small values of g, negative for $g = \frac{1}{2}$ and that $\varphi_1(g)$ has at most two extrema in the interval $(0,\frac{1}{2})$. Hence $\varphi_1(g)$ has exactly two extrema and exactly one zero, g_0, say, in the interval $0 < g < \frac{1}{2}$. Moreover $\varphi_1(g)$ is positive for $0 < g < g_0$ and negative for $g_0 < g \leq \frac{1}{2}$. The function $\varphi_1(g)$ being the derivative of $\varphi(g)$, apart from a positive factor (for $g \neq 0$), the proof is completed.